New $\mathcal{N} = 1$ Superconformal Field Theories in Four Dimensions from D-brane Probes

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We present several new examples of nontrivial $4d \mathcal{N} = 1$ superconformal field theories. Some of these theories exhibit exotic global symmetries, including non-simply laced groups (such as F_4). They are obtained by studying threebrane probes in F-theory compactifications on elliptic Calabi-Yau threefolds. The geometry of the compactification encodes in a simple way the behavior of the gauge coupling and the Kähler potential on the Coulomb branch of these theories.

10/96

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1. Introduction

Recent developments have led to dramatic improvements in our understanding of quantum field theory and string theory. For example, inspired by the beautiful work of [1,2,3], the study of quantum field theories on brane probes has led to the discovery of a new and surprising set of exotic fixed points of the renormalization group in three, four, five, and six dimensions [4,5]. For some of these fixed points no Lagrangian is known that flows to them in the infrared, suggesting that other methods are necessary to study general field theories (and, in particular, conformal field theories). The theories studied in [4,5] have the equivalent of 4d $\mathcal{N}=2$ supersymmetry. In this paper we present a new class of renormalization group fixed points with 4d $\mathcal{N}=1$ superconformal symmetry. These are obtained as the effective worldvolume theory on threebrane probes in six-dimensional $\mathcal{N}=1$ string compactifications.

The known examples of $\mathcal{N}=1$ superconformal theories fall into two classes. One class consists of finite theories, in which the coupling parametrizes a line of fixed points [6]. For these there is a Lagrangian description of the fixed point theory (which is useful at least at weak coupling). One of the new superconformal field theories we discuss is of this type. Another class which has been investigated in detail in the past two years consists of the infrared limit of theories like $\mathcal{N}=1$ $SU(N_c)$ gauge theories with $\frac{3}{2}N_c < N_f < 3N_c$ flavors of quarks [7]. There are two known asymptotically free quantum field theories which flow to these nontrivial fixed points in the IR. These theories exist for a fixed (unknown) value of the IR gauge coupling. Most of the theories we find exemplify a third class of $\mathcal{N}=1$ superconformal theories: we find nontrivial fixed points with exotic global symmetries (including F_4) at fixed values of the coupling. There are no known candidates for Lagrangian field theories which flow to these theories in the infrared. Under relevant perturbations, they flow to SU(2) or U(1) gauge theories with known matter content. This is similar in many ways to the theories recently discussed in [4].

The six-dimensional theories of interest are obtained by compactification of the heterotic string on K3 with 12+n instantons in one E_8 factor and 12-n in the other. This is dual [8] to F-theory compactified on the Calabi-Yau manifold which is an elliptic fibration over the Hirzebruch surface F_n . When the K3 on the heterotic side is an elliptic fibration over \mathbf{P}^1 , we can consider a threebrane obtained by wrapping the fivebrane on the T^2 fiber. The quantum field theory on this threebrane has an SU(2) gauge symmetry with an adjoint scalar X and an additional scalar field which parametrizes the position on the base

 $\mathbf{P^1}$. On the F-theory side, this is dual to a Dirichlet threebrane probe whose moduli space includes the F_n surface. The threebrane is the natural probe to consider in F-theory, since it is invariant under the $SL(2, \mathbf{Z})$ duality symmetry of type IIB string theory. Viewing F_n as a $\mathbf{P^1}$ bundle over $\mathbf{P^1}$, $\operatorname{tr}(X^2)$ of the heterotic theory maps to the position on the $\mathbf{P^1}$ fiber while the additional scalar maps to the position on the base $\mathbf{P^1}$.

There is a simple relation between the spacetime theory and the worldvolume quantum field theory of a D-brane probe. Fields in the spacetime theory become parameters in the brane theory. The spacetime gauge symmetry which is apparent locally to the probe is (part of) the global symmetry on the brane. The complex structure τ of the elliptic fiber in F-theory is the infrared gauge coupling on the threebrane probe. The metric of the compactification manifold gives the metric on the Coulomb branch of the field theory on the probe. This is part of a general story in which short distance effects in spacetime are translated (via a relation between distances and masses) to long distance effects on D-brane probes. These facts allow us to identify nontrivial interacting $\mathcal{N}=1$ fixed points, which in some cases possess exceptional global symmetry, by moving the probe to appropriate points in spacetime. For generic points on the base \mathbf{P}^{1} , the models reduce in the infrared to 4d $\mathcal{N}=2$ theories studied on threebrane probes in the eight-dimensional spacetime theory [2,3]. However, at special points on this \mathbf{P}^1 having to do with the nontrivial fibration, the brane theory changes and exhibits properties indicating that it has become a nontrivial $\mathcal{N}=1$ superconformal theory. The consistency of our results suggests that it is valid to use threebranes as probes in six dimensions, despite the fact that they induce a deficit angle in the noncompact space. This could be related to the fact that our analysis uses only local properties of the Calabi-Yau manifold, and could in principle be performed also when this manifold is not compact.

Compactifications of the heterotic string on K3 yield an intricate web of 6d $\mathcal{N}=1$ theories, the F-theory description of which was worked out in [8,9,10]. Because of the relations reviewed in the previous paragraph, Higgsing in spacetime decreases the global symmetry on the brane. This partially determines the operator content and couplings present in the worldvolume theory. In section two, we review the relevant spacetime theories. In section three, we discuss theories with SO(n) global symmetry, for which we have a Lagrangian description. Theories with exceptional global symmetry, and other theories for which we do not have a Lagrangian description, are covered in section four. Finally, we present our conclusions and possible generalizations in section five.

2. Three-Brane Probes and Nontrivial Fixed Points

A fairly detailed map from the moduli space of the $E_8 \times E_8$ heterotic string on K3 with (12 + n, 12 - n) instantons in the two E_8 s to the complex moduli of the elliptic fibration over F_n was given in [10]. Following the notation of that paper, we denote the coordinate on the $\mathbf{P^1}$ base of the F_n as z_2 and the coordinate on the $\mathbf{P^1}$ fiber as z_1 . The unbroken subgroups $H_{1,2}$ of the two E_8 gauge groups of the heterotic theory are localized on the divisors at $z_1 = 0$ and $z_1 = \infty$. If we move the threebrane probe to either of these divisors, we obtain (at least) an $H_{1,2}$ global symmetry in the long-distance quantum field theory on the probe.

In the F-theory description, one obtains the unbroken spacetime gauge symmetry by fibering an A - D - E singularity over the z_2 sphere. At generic points on the z_2 plane, the brane probe flows in the infrared to an $\mathcal{N} = 2$ superconformal theory with A - D - E global symmetry, and the additional spacetime sphere has no effect. At special points on the z_2 plane, the singularity becomes worse. There are two things which can happen at such special points:

- (1) The A D E gauge group undergoes an outer automorphism as we go around the special point [9]. In this case, the gauge group is broken from the simply laced A D E group to a non-simply laced group Sp(k), SO(2k + 1), F_4 , or G_2 . At this type of special point, we will find that in some cases the probe theory flows instead to an $\mathcal{N} = 1$ superconformal theory with a non-simply laced global symmetry group.
- (2) Charged matter of the spacetime gauge group is localized at the special point. In this case the theory on the brane probe sees extra light particles as one approaches the special point. If the light particles in the probe theory have mutually non-local charges, one again obtains a nontrivial superconformal field theory. We find evidence that in cases where whole spacetime hypermultiplets are localized at a point the brane probe sees an $\mathcal{N}=2$ superconformal theory in the infrared, while at "half hypermultiplet" points it sees an $\mathcal{N}=1$ superconformal theory. Examples of both types of localization are known [8,9,10]. Note that in the cases where the spacetime gauge group is non-simply laced, the matter is not necessarily localized in this manner but there are interesting points of type (1).

Under a relevant perturbation, the new fixed points we find flow to SU(2) or U(1) gauge theories with Coulomb phases (which correspond to generic points on D_N and A_N singular curves). The F-theory description of the spacetime theory involves an elliptically fibered Calabi-Yau manifold defined by an equation of the form

$$y^{2} = x^{3} + xf(z_{1}, z_{2}) + g(z_{1}, z_{2}).$$
(2.1)

Expanding (2.1) around singular points directly provides the "Seiberg-Witten curve" determining the gauge coupling function on the Coulomb branch for the field theory on our brane probe [11,12,13]. Note that our theories always have Coulomb branches, since at a generic point on the F_n the worldvolume theory on the brane probe is a U(1) gauge theory. The singular points in the elliptic fibration (near which the field theory on the brane probe has interesting behavior) are points where the discriminant

$$\Delta = 4f^3 + 27g^2 \tag{2.2}$$

vanishes. Then, using

$$j(\tau) \sim \frac{f^3}{\Delta} \tag{2.3}$$

which gives the complex structure of the fiber torus, we can determine the gauge coupling τ on the Coulomb branch around our new fixed points.

In the F-theory compactifications we are studying, one obtains chains of different models related by Higgsing in spacetime. This is reflected in Diagram 1, which is borrowed from [10].

Diagram 1: Phase Diagram

In the context of our brane probe field theories, the *gauge* symmetries in spacetime become *global* symmetries of the infrared theory on the probe. The arrows (reflecting spacetime

Higgsing) become flows between the worldvolume field theories under relevant perturbations. When the Higgsing in spacetime breaking gauge group G to gauge group H is accomplished by giving VEVs to fields in the R representation of G, we expect that these spacetime fields act as parameters in the worldvolume field theory coupled to operators in the \bar{R} representation. Turning on these parameters introduces a relevant perturbation under which the theory flows from the fixed point with global symmetry containing G to a new one with global symmetry containing H in the infrared.

3. Lagrangian Field Theories and Fixed Lines from D_N Singularities

3.1. SO(2N) Theories

In order to illustrate our methods, it is useful to begin with examples for which we can propose a Lagrangian description. Most of these theories flow in the IR to free theories, but they may reached by relevant perturbations from our nontrivial fixed points.

Let us begin with a D_N singularity in the "upper" \mathbf{P}^1 ($N \geq 4$). According to [3], the field theory of a threebrane probe near such a singularity is an $\mathcal{N}=2$ supersymmetric SU(2) gauge theory with N quark flavors, which has an SO(2N) global symmetry. The coordinate z_1 of the threebrane on the \mathbf{P}^1 corresponds to the gauge invariant field $u=\operatorname{tr}(X^2)$ (where X is the adjoint scalar in the SU(2) gauge multiplet). The D_N singularity may be interpreted as a \mathbf{Z}_2 orientifold point with N mutually local sevenbranes, and the N quarks then arise from strings between the threebrane and these sevenbranes [2,3]. In eight dimensions there is (in spacetime) an SO(2N) adjoint scalar which is part of the vector multiplet (corresponding to the location of the sevenbranes on the \mathbf{P}^1), and on the brane probe this corresponds to a mass matrix for the quarks.

When we fiber this singularity over an additional \mathbf{P}^1 , the SO(2N) adjoint scalars are lifted, though we expect them to still be visible to a threebrane probe when it is far from any singular fibers, since in the IR the probe will just see the eight dimensional theory. At special points on the bottom \mathbf{P}^1 the singularity becomes worse. In the F-theory description, these points correspond to additional sevenbranes intersecting the original sevenbranes we had (corresponding to the D_N singularity). When the intersecting sevenbranes are mutually local to the N sevenbranes discussed above, we get SO(2N) charged matter in

the 2N representation.¹ In other cases more interesting representations may appear. We will discuss these in the next section.

Particles which arise from strings between the threebrane and a (p,q) sevenbrane have electric charge p and magnetic charge q in the threebrane field theory, so we can expect to have a local Lagrangian description (which is different from the eight dimensional description) only at points where we have massless 2N fields localized in spacetime. At such a point (which we choose to be at $z_2 = 0$) the spacetime 2Ns should correspond to parameters in the 2N representation of SO(2N) in the worldvolume theory, and we expect to have additional massless fields coming from the strings between the threebrane and the additional sevenbranes. Thus, the Lagrangian for a threebrane near a massless vector point should be of the form

$$W = Q^a X Q^a + z_2 q^1 q^2 + \lambda_{ij} q^i X q^j + m^a q^1 Q^a,$$
(3.1)

where Q^a $(a = 1, \dots, 2N)$, q^i (i = 1, 2) are SU(2) doublets, z_2 is a worldvolume singlet field (corresponding to the location of the brane in the "bottom" \mathbf{P}^1), and m^a is the spacetime $2\mathbf{N}$. The $\lambda q X q$ term can be understood as follows. The discriminant of the elliptic fiber near the singularity looks like

$$\Delta = z_1^{N+2} (\alpha z_2^2 + \beta z_1 + o(z_1^2)) \tag{3.2}$$

where α and β are functions of z_2 which go to constants as $z_2 \to 0.2$ Zeroes of the discriminant correspond to sevenbranes which generically give rise to extra massless fields on the threebrane; from (3.2) we see that this occurs when $\alpha z_2^2 + \beta z_1 = 0$. This is reproduced by our Lagrangian if we identify $\det(\lambda)$ with β/α . The last term in (3.1) reflects the Higgs mechanism in spacetime, which corresponds to global symmetry breaking on the brane. We have chosen q^1 to be the field that couples to m^a by an SO(2) rotation of the q fields. When $m^{2N} \neq 0$, we see that the global SO(2N) symmetry is explicitly broken to SO(2N-1) as required.

The Lagrangian written above (and any others appearing in this paper) should be understood as giving a "phenomenological" description of the low energy physics on the

¹ When monodromies on the bottom \mathbf{P}^1 break SO(2N) to a smaller group, this localization of matter does not occur. Such cases will be discussed in the following subsection.

Note that the formula given for Δ on the SO(8) locus in §4.7 of [10] is incorrect. The correct formula is $\Delta \sim z_1^6 (f_{2n+8}^2 q_{n+4}^2 + o(z_1))$ which agrees with (3.2).

brane probe when it is near the singularity. We expect the sort of terms we write, as well as terms involving higher orders in z_1 and z_2 which we suppress, to arise from interactions between the (fundamental) strings ending on the probe and on the spacetime sevenbranes. At weak coupling it should be possible (for the theories discussed in this section) to actually compute these terms from string theory.

From the expression for the j function (2.3), and from the known form of the Seiberg-Witten curve, we see that for these theories $\tau \sim \log(z_1^{N-4}z_2^2)$, which goes to $i\infty$ at the singularity $(z_1 = z_2 = m^a = 0)$. This indicates that, as expected, these theories are free in the infrared. In particular, the theory (3.1) flows (for $N \geq 4$) to a free $\mathcal{N} = 2$ theory with SO(2N+2) global symmetry. This is in accord with the picture of Katz and Vafa [14], which describes the appearance of 2N matter in the six dimensional theory by starting with an SO(2N+2) theory in eight dimensions and turning on a z_2 dependent adjoint VEV which breaks SO(2N+2) to SO(2N). Both z_2 and the mass parameters m^a become part of the SO(2N+2) adjoint parameter of this infrared theory. Turning on either z_2 or m is equivalent to turning on the adjoint in the SO(2N+2) theory, and causes us to flow to the SO(2N) $\mathcal{N}=2$ theory. When more than one **2N** field is present, for generic values of z_2 the IR theory will have an SO(2N) adjoint parameter that includes a contribution of the form $h(z_2)m_i^{[a}m_j^{b]}$ where m_i and m_j are any two spacetime **2N** fields and $h(z_2)$ is some function of z_2 .³ Turning on two of these spacetime fields breaks the spacetime gauge group to SO(2N-2), and this is realized by the adjoint breaking in the worldvolume $\mathcal{N}=2$ superconformal theory.

There are special loci in the moduli space of the compactification where k spacetime vector points come together. In this case the discriminant goes like

$$\Delta \sim z_1^{N+2} (\alpha z_2^{2k} + \beta z_1),$$
 (3.3)

and we see that the point with extra massless matter occurs at $\alpha z_2^{2k} + \beta z_1 = 0$. The total number of spacetime **2N** fields is n + 2N - 4, suggesting that the Lagrangian (3.1) should be generalized to

$$W = Q^{a}XQ^{a} + P_{n+2N-4}(z_{2})q^{1}q^{2} + \lambda_{ij}q^{i}Xq^{j} + \sum_{\alpha=1}^{n+2N-4} (p_{\alpha}(z_{2})m_{\alpha}^{a}q^{1} + \tilde{p}_{\alpha}(z_{2})\tilde{m}_{\alpha}^{a}q^{2})Q^{a}$$
(3.4)

³ The worldvolume Lagrangian description of the theories with multiple **2N**s in spacetime is provided below.

where P_{n+2N-4} is a polynomial of degree n+2N-4 in z_2 whose zeros correspond to the location of the spacetime matter fields [10], p_{α} and \tilde{p}_{α} are polynomials in z_2 , and m_{α}^a and \tilde{m}_{α}^a are the two complex scalars in the hypermultiplet corresponding to the α th massless **2N** field. This has two effects. First, when we bring k zeroes of P_{n+2N-4} together, we recover the behavior determined by the discriminant (3.3). Also, we have incorporated the fact that when k spacetime vector points converge, we should have k spacetime vectors which couple to naively distinct operators $z_2^i q^1 Q^a$, $z_2^i q^2 Q^a$ (for i < k these are in the chiral ring). In this case the worldvolume theories corresponding to different values of k all flow to the same (free) theory in the infrared, in which z_2 is a free field.

3.2. SO(2N-1) Theories

The generic theory with a D_N singularity at a point on the top \mathbf{P}^1 actually yields SO(2N-1) gauge symmetry in spacetime [9,10].⁴ At special points on the bottom \mathbf{P}^1 there is monodromy implementing an outer automorphism on the D_N Dynkin diagram, yielding a B_{N-1} diagram.

In F-theory, the locus on which there is no such monodromy has discriminant

$$\Delta \sim z_1^{N+2} (q_{n+2N-4}(z_2)^2 + o(z_1)). \tag{3.5}$$

Breaking SO(2N) to SO(2N-1) involves generalizing q_{n+2N-4}^2 to a generic polynomial P of degree 2n + 4N - 8 in z_2 . This means that, for small values of m^a , the special $2\mathbb{N}$ points on the z_2 plane each split into two points as one deforms from the SO(2N) to the SO(2N-1) theory.

The monodromy is localized at the zeroes of P on the z_2 plane. Therefore, at generic points, one expects to flow in the IR to a theory with SO(2N) symmetry, while at the zeroes of P sometimes the IR theory will exhibit only SO(2N-1) symmetry. From the Lagrangian (3.4) we see that if m^a is nonzero, the SO(2N) symmetry is explicitly broken to SO(2N-1). However, there are still 2N massless quarks in the theory, and the theory will generically flow to the $\mathcal{N}=2$ supersymmetric theory with a full SO(2N) global symmetry, which is an IR-stable fixed line [6]. The coupling of the extra massless quark to X generically flows to the $\mathcal{N}=2$ value in the infrared.

⁴ Except in the case N=4 where G_2 is the generic symmetry and SO(7) is a special case, or in cases where the spacetime field content is not compatible with Higgsing to SO(2N-1).

For N > 4 the theory is free in the IR and the meaning of this statement is not clear. For N = 4, the theory is finite at generic points, but still becomes free at the special points we are discussing, as we will see below. For N = 3 we find interesting behavior at the monodromy points, which we will discuss in §3.4.

Let us show how this behavior is reproduced by the Lagrangian (3.4). Consider the case N=4 and $m\neq 0$. Then, integrating out the massive quarks, we obtain the effective superpotential

$$W = \sum_{a=1}^{7} Q^a X Q^a + (z_2^2 + \lambda_{12}^2 z_1 + \lambda_{22} m^2) q X q$$
(3.6)

where q is an appropriate combination of q^2 and Q^8 (i.e. the combination that remains massless). Now, we see that for nonzero m, there are two different types of behavior. depending on the value of z_2 . For generic z_2 , the qXq term is nonzero, and the theory will flow to the $\mathcal{N}=2$ SO(8) theory in the infrared. However, for z_2 a solution of $z_2^2 + \lambda_{22}m^2 = 0$, the bare qXq coupling vanishes and it cannot be perturbatively generated. Thus, for these two special points on the z_2 plane, one might expect to flow to a fixed point with SO(7) global symmetry and $\mathcal{N}=1$ supersymmetry. In fact, the theory flows to a free theory in the infrared, as can be seen by examining the behavior of the j function (2.3) at the special points. This can also be seen by looking at the beta functions for the couplings in (3.6) at the special points. For non-zero $z_1 \sim \operatorname{tr}(X^2)$, the Q quarks are all massive (as in the SW $N_f = 4$ theory), but for z_2 such that the coefficient of the qXqterm vanishes, q will remain massless. In this case we will flow to an $\mathcal{N}=2$ U(1) gauge theory with one massless electron. This is consistent with the form of the discriminant of the SW curve around the SO(7) monodromy points. Note that the original singularity at the 2N point has split as we expected, and that our superpotential explicitly shows that the matter on the threebrane is no longer localized at a specific value of z_2 , just like the spacetime matter in this case.

The local behavior of the discriminant at points at which a monodromy breaks SO(8) to G_2 is exactly the same as the behavior at SO(7) monodromy points. Thus, even though we have no local Lagrangian description of these theories, we expect that they will also flow to free theories in the IR, though it is not clear which variables ("electric" or "magnetic") should become free in this case.

3.3. $SU(4) \simeq SO(6)$

So far we have only used the Lagrangian formulation for cases where there were no quantum corrections to the classical moduli space. But, as discussed by Sen [2], F-theory also correctly describes the quantum corrections to the moduli space when quark masses are added to the SO(8) theory. In particular, giving a mass to one quark leads to a theory with SO(6) global symmetry. This is the $N_f = 3$ case studied by Seiberg and Witten [12]. The quantum moduli space of this theory has two singularities, one of which corresponds to a massless 4 and the other to a massless singlet. In F-theory we find an A_3 singularity at $z_1 = 0$, and we identify this with the Seiberg-Witten 4 point. The field theory of the probe near such a singularity is an $\mathcal{N} = 2 U(1)$ gauge theory with four massless electrons.

Special points on the z_2 plane are associated with spacetime fields in the **4** or **6** representation of SO(6). Consider the theory near a **6** point: we expect the classical superpotential to be given by (3.1). When $m = z_2 = 0$, there are 4 massless quark hypermultiplets, and the theory will flow in the IR to the SO(8) $\mathcal{N} = 2$ theory (with a finite value of the gauge coupling). At this fixed point, z_2 is the mass of one flavor. The curve around this point is exactly the same as the Seiberg-Witten curve for $N_f = 4$ with one small mass turned on [12]. In both cases the order 6 singularity at the origin of moduli space splits into one singularity of order 4 (which we can keep at $u \equiv z_1 = 0$) and two singularities of order 1 which occur at $u \sim z_2^2$.

3.4.
$$SO(5) \simeq Sp(2)$$
 Fixed Lines

As in the discussion of §3.2, we can turn on a VEV for the spacetime **6** field. Again, the F-theory picture suggests the existence of two special values of z_2 (for $z_1 = 0$) for which the theory would flow to a theory with SO(5) global symmetry, and this is supported by a classical analysis of our superpotential (3.1). After integrating out the massive fields we find that near such a special value of z_2 the SO(5) theory is described by the superpotential

$$W \sim \sum_{a=1}^{5} Q^a X Q^a + (z_2 + h z_1) q X q.$$
 (3.7)

In this case our theory is an asymptotically free theory (with one adjoint chiral multiplet and 6 doublets), so quantum corrections are expected to be important (as they were in the SO(6) theory).

The form of the j function (2.3) as one approaches a monodromy point at $z_2 = 0$ is as follows:

$$j(\tau) \sim \frac{(b_1 z_1^2 + b_2 z_1 z_2 + b_3 z_2^2)^3}{z_1^4 (c_1 z_1^2 + c_2 z_1 z_2 + c_3 z_2^2)}.$$
 (3.8)

Approaching the fixed point with a given ratio of z_1 to z_2 , we see that the limiting value of τ varies as a function of the moduli b_i, c_i (which are related to the couplings in the superpotential (3.7)). This suggests the existence of a fixed line with every possible value of $j(\tau)$. Indeed, this behavior of the j function near the singularity is similar to that of the j function near the singularity corresponding to the $\mathcal{N} = 2$ SO(8) theory with a small mass for one quark, which was discussed in §3.3. In that case we know there is a fixed line. From (3.8) we see that there is an ambiguity in the limiting value of τ coming from the choice of z_1/z_2 as we approach the singularity. This reflects the fact that z_1 and z_2 control the masses of mutually nonlocal states, and it prevents us from reading off the value of $j(\tau)$ at the singularity.

The superpotential for these theories at the monodromy points looks like

$$W = \lambda \sum_{a=1}^{5} Q^{a} X Q^{a} + h \operatorname{tr}(X^{2}) q X q.$$
 (3.9)

We can check for the plausibility of the existence of a fixed line along the lines of [6]. Computing the beta functions for the gauge coupling g, and for the couplings h and λ , one finds that they are proportional to the scaling coefficients

$$A_q = 2 + 5\gamma_Q + \gamma_q + 4\gamma_X \tag{3.10}$$

$$A_{\lambda} = \frac{1}{2}\gamma_X + \gamma_Q \tag{3.11}$$

$$A_h = 2 + \frac{3}{2}\gamma_X + \gamma_q. {(3.12)}$$

In particular, we find that

$$A_g = A_h + 5A_\lambda. (3.13)$$

From (3.13) we see that there are only two independent scaling coefficients for three variables, so generically we expect that a fixed line should exist in (g, h, λ) space. This is consistent with the computation of $j(\tau)$ from F-theory, and lends credence to our use of the superpotential (3.9) and in particular to our assumption that the $htr(X^2)qXq$ term plays an important role in the infrared.

Note that the (3.9) fixed line does not pass through any weak coupling regime where a perturbative analysis would be reliable. The behavior of the j function suggests that the fixed line should exist for any value of $j(\tau)$, but this does not necessarily mean that it passes through a region of weak "electric" coupling – perhaps when $j(\tau)$ is large it is actually the "magnetic" gauge coupling that is weak. In any case, for λ , h small the theory certainly flows to strong gauge coupling.

We have found that at $z_1 = z_2 = 0$ the theory flows to a superconformal fixed point. Next, we should try to compute the dimensions of the operators at this point. As in [15,16], we can compute ratios of dimensions of operators directly from the elliptic curve describing the behavior of the gauge coupling near the fixed point. In the SO(5) case, this curve looks like

$$y^{2} = x^{3} + x(z_{2}^{2} + z_{2}z_{1} + z_{1}^{2} + \cdots) + (z_{2}^{3} + z_{2}^{2}z_{1} + z_{2}z_{1}^{2} + z_{1}^{3} + \cdots)$$
(3.14)

where \cdots denote terms of higher order in z_1 and/or z_2 , and constant coefficients for all terms are suppressed. Taking the leading terms in the curve to have equal dimension, we find that $[z_1] = [z_2]$. In the next paragraph, we will normalize the dimensions of the fields by using information present in the string theory. Even before doing so, we can check whether the fixed point is $\mathcal{N}=2$ or $\mathcal{N}=1$ superconformal. If the theory does have $\mathcal{N}=2$ supersymmetry, we can determine the dimensions (as in [15,16]) by using the fact that $\int u \frac{dx}{u}$ should have dimension one (since it gives the mass of BPS states). In our case, we identify z_1 with u, and find that this assumption leads to $[z_1] = [z_2] = 2$. While this is the natural dimension for z_1 in an $\mathcal{N}=2$ theory (since $z_1\sim \operatorname{tr}(X^2)$ and X is in a vector multiplet), it seems inconsistent with $\mathcal{N}=2$ supersymmetry to have $[z_2]=2$. This is because z_2 has no $\mathcal{N}=2$ superpartner with the same dimension, and hence would have to behave as a decoupled free field in the infrared (i.e., have $[z_2] = 1$) for the fixed point to have $\mathcal{N}=2$ supersymmetry. Thus, the $\mathcal{N}=2$ assumption seems to lead to a contradiction, and we are led to believe that the new superconformal theory we have found actually has only $\mathcal{N}=1$ supersymmetry. It is clear that the global symmetry of this superconformal theory is at least SO(5). In principle, the global symmetry might be enhanced in the infrared, but we see no reason for this to happen in this case.

In fact, using the information present in F-theory we can even precisely normalize the dimensions $[z_1], [z_2]$ at our $\mathcal{N} = 1$ fixed point.⁵ The Calabi-Yau M, which we take to be

⁵ We thank C. Vafa for suggesting this to us.

elliptically fibered with base F_n , has a holomorphic three-form Ω . It follows from special geometry that, if the elliptic fiber is taken to have constant volume,

$$\int_{M} \Omega \wedge \overline{\Omega} \sim Vol(M) \sim Vol(T^{2}) \times Vol(F_{n})$$
(3.15)

and that the volume form on the Coulomb branch of the field theory (which we identify with the base F_n) is given by

$$\int_{T^2} \Omega \wedge \overline{\Omega} \ . \tag{3.16}$$

So, holding the volume of the T^2 fixed (as F-theory instructs anyway), and performing a scale transformation in the field theory, we expect Ω to have $[\Omega] = 2$. A general formula for Ω given a threefold defined by an equation $W(x, y, z_1, z_2) = 0$ is

$$\Omega \sim \frac{dx \wedge dz_1 \wedge dz_2}{\frac{\partial W}{\partial y}} \ . \tag{3.17}$$

From (3.14), we see that Ω is given in a neighborhood of the singularity as

$$\Omega \sim \frac{dx \wedge dz_1 \wedge dz_2}{y} \ . \tag{3.18}$$

Hence,

$$[z_1] + [z_2] + [x] - [y] = 2,$$
 (3.19)

and using the known ratios between the dimensions we find that

$$[z_1] = [z_2] = \frac{4}{3} . (3.20)$$

Note that if there is a fixed line, the dimensions do not vary along it. This might indicate that the photon, whose gauge coupling changes along the fixed line, decouples from the nontrivial conformal theory.

The realization of our quantum field theory on a threebrane in F-theory enables us to compute the Kähler potential even though the theory only has $\mathcal{N}=1$ supersymmetry. The metric on the Coulomb branch of our field theory is just the restriction of the elliptic Calabi-Yau metric to the F_n base. Although the Calabi-Yau metric is not known in general, it can be computed in the vicinity of singularities (for example near conifold singularities the metric is given in [17]). From the conformal field theory point of view it is only this information that is relevant in any case. The global form of the metric depends on massive

degrees of freedom that decouple from the fixed point theory. It would be interesting to try to understand the appearance of (3.18) directly in the field theory.

As described above, we can flow to the new SO(5) superconformal theory by starting with the SO(6) theory at a massless 6 point, and turning on z_2 and m with an appropriate ratio between them. At $z_2 = m = 0$, that theory flows in the infrared to the $\mathcal{N} = 2$ SU(2) gauge theory with $N_f = 4$, which has an SO(8) global symmetry and a parameter in the adjoint of SO(8) (corresponding to a mass matrix for the quarks). The $N_f = 4$ theory also has (in general) additional parameters, such as the Yukawa coupling appearing in front of a term $Q^a X Q^b$ in the superpotential, which is in the symmetric tensor representation of SO(8). In terms of the infrared conformal theory, turning on z_2 and/or m corresponds to turning on both adjoint perturbations (which preserve $\mathcal{N} = 2$) and symmetric tensor perturbations (which break $\mathcal{N} = 2$). For generic values of the parameters the theory will flow back to an $\mathcal{N} = 2$ theory in the infrared, despite the explicit breaking we turned on. However, for special values of z_2 and m (as described above), we will actually flow to the $\mathcal{N} = 1$ SO(5) theory.

We can also discuss what happens when we bring k monodromy points together. For even k, we expect at least the SO(6) symmetry to be restored in this case (since each point corresponds to a \mathbb{Z}_2 monodromy), and presumably the behavior is similar to the theories discussed in the previous subsection (i.e. the theory flows to the $\mathcal{N}=2$ theory with SO(8) global symmetry). For odd k the behavior of the theory is not a priori clear, and we have not been able to uniquely identify a Lagrangian that will flow to these points. Performing the calculation (3.19) for this case yields $[z_1] = \frac{4k}{k+2}$ and $[z_2] = \frac{4}{k+2}$. For k=2 this is consistent with the expectation that the theory flows to the $\mathcal{N}=2$ SO(8) theory. For higher k, the naive dimension for z_2 is less than one. This would seem to indicate that the theory is nonunitary, since the superconformal algebra requires scalar fields in a unitary theory to have dimension ≥ 1 .

However, as discussed in [18,15], the proper interpretation of this is that z_2 is actually a free field which enters the Lagrangian as part of a "dangerously irrelevant" operator. Specifically, the Lagrangian includes terms of the form $\mu^{-\delta} \int d^2\theta z_2 \mathcal{O}$, where the dimension of the operator \mathcal{O} satisfies $3 > [\mathcal{O}] = 2 + \delta > 2$ and z_2 is a free field. At the fixed point, this operator is irrelevant and flows away in the infrared. However, upon giving a vacuum expectation value to z_2 , the resulting deformation $\mu^{-\delta} \int d^2\theta \langle z_2 \rangle \mathcal{O}$ becomes relevant. Now we can identify $[z_2]$ computed from the curve with the dimension of $\mu^{-\delta} \langle z_2 \rangle$.

Going back to our example, we learn that for k > 2 the physical dimension of z_2 is one, and it decouples from the theory. Plugging this back into (3.19), we recover the formula $[z_1] + [x] - [y] = 1$, which is just the $\mathcal{N} = 2$ formula. This is not surprising since when z_2 decouples we expect the theory to reduce to an appropriate $\mathcal{N} = 2$ brane theory in eight dimensions. Calculating the dimension of z_1 , we find that for all k > 1, $[z_1] = 2$. For even k, this is consistent with our expectation that these theories flow to the $\mathcal{N} = 2$ SO(8) theory. For odd k the situation is less clear: the dimensions are those of the $\mathcal{N} = 2$ SO(8) theory, but on the other hand we would expect the monodromy breaking SO(6) to SO(5) to reduce the global symmetry here just as in the k = 1 case (3.7).

4. Exceptional Groups and Other Nontrivial Fixed Points

In this section we will discuss fixed points for which we have no local Lagrangian description. For threebrane probes at E_n singularities no Lagrangian description is known already in the eight dimensional case, and we will discuss the $\mathcal{N}=1$ superconformal theories related to these singularities in §4.1 and §4.2. In §4.3 we discuss additional examples of such theories, which are obtained by probing points at which spacetime fields in the spinor of SO(2N) are localized. In analogy to the way **2N**s in spacetime coupled to operators in the **2N** representation in the worldvolume, we expect couplings of spacetime fields in the spinor of SO(2N) to worldvolume operators in the spinor of SO(2N).

In the SO(8) case there is a triality symmetry between vectors and the two types of spinor representations. Thus, in this case, when we approach a spinor point an additional "magnetic" quark becomes massless and the superpotential in terms of magnetic variables is the same as above (3.4). In this case of course these points will just be weakly coupled in terms of the "magnetic" variables. In the more general situation, for example SO(10), we still expect to find additional massless magnetically charged particles at these points. In this case the Q^a , which tranform in the $\bf 10$ representation, are necessarily "electric" variables, and no local Lagrangian description including both "electrically" and "magnetically" charged particles is known. This is similar to the fixed points discussed in [19,15]. In general, when the threebrane approaches a point where a spacetime field in the representation $\bf R$ is localized, additional degrees of freedom will come down which are parts of composite operators on the worldvolume which are in the $\bf \bar{R}$ representation. We expect the spacetime fields, which are parameters on the brane, to couple to the worldvolume $\bf \bar{R}$ s. Then, when the spacetime fields obtain VEVs, the global symmetry breaks appropriately as discussed above.

4.1. Evidence for F_4 Theory

Let us begin by considering what happens when the upper \mathbf{P}^1 develops an E_6 singularity. The curve in this case is of the form

$$y^{2} = x^{3} + (z_{1}^{3} f_{n+8}(z_{2}) + \cdots)x + z_{1}^{4} g_{2n+12}(z_{2}) + \cdots$$

$$(4.1)$$

and the discriminant looks like

$$\Delta \sim z_1^8 g_{2n+12}^2 + \cdots$$
 (4.2)

In (4.1) and (4.2), \cdots represents terms of higher order in z_1 . For generic g_{2n+12} , the spacetime gauge symmetry is actually F_4 . This occurs, as explained in [9] and [10], because of a \mathbb{Z}_2 monodromy about zeroes of g_{2n+12} on the lower \mathbb{P}^1 . When $g_{2n+12} \sim q_{n+6}^2$, the monodromies vanish and the spacetime gauge symmetry is the full E_6 . In both cases, at generic values of z_2 , the threebrane theory flows to the $\mathcal{N}=2$ E_6 theory of [4,5] by the adiabatic argument.

Let us look at what happens when the probe approaches a zero of g_{2n+12} of multiplicity k. Consider first the cases with k odd. Going around such points induces a monodromy on the E_6 Dynkin diagram which breaks $E_6 \to F_4$. Therefore we expect that F_4 is the global symmetry in the infrared. This is analogous to the SO(5) global symmetry we found in §3.4, when we had a Lagrangian description.

Given the curve (4.1), we can compute the dimensions of z_1 and z_2 at the fixed point as a function of k, as in the previous section. To do this, let us take all leading terms in the curve to have the same dimension. This leads to the result

$$[z_1] = \frac{8k}{k+2}, \ [z_2] = \frac{4}{k+2}.$$
 (4.3)

For k = 1, we get a nontrivial interacting fixed point which we expect to have F_4 global symmetry. This theory cannot have extended supersymmetry since $[z_2] = 4/3$. Computing the value of the U(1) gauge coupling constant on the Coulomb branch near our singularity using the j-function (2.3), we find $\tau = i$.

For even k, we can describe this singularity by starting with an E_7 singularity in the upper \mathbf{P}^1 and giving the adjoint a z_2 -dependent VEV [14], suggesting that the theory flows to the E_7 $\mathcal{N}=2$ theory in the infrared. The dimensions we find are consistent with this. From this theory we can flow to the k=1 theory by turning on a **27** field in spacetime,

which corresponds to splitting the two zeros of g_{2n+12} . In the $\mathcal{N}=2$ superconformal theory this would correspond to turning on perturbations which break $\mathcal{N}=2$ (these exist in addition to the adjoint perturbations that preserve $\mathcal{N}=2$) and which break E_7 to F_4 . This is analogous to the similar situation discussed in §3.4. For odd k>1, the dimensions are the same as for even k, but, as in the SO(5) case, the global symmetry (and the number of supersymmetries) is not clear.

4.2. New E_7 Fixed Point

Similarly, we can go to the E_7 locus on which the curve looks like

$$y^{2} = x^{3} + (z_{1}^{3} f_{n+8}(z_{2}) + \cdots)x + z_{1}^{5} g_{n+12}(z_{2}) + \cdots,$$

$$(4.4)$$

and the discriminant is

$$\Delta \sim z_1^9 (f_{n+8}^3(z_2) + o(z_1)). \tag{4.5}$$

At generic points in the z_2 plane, the theory flows (by the adiabatic argument) to the previously studied $\mathcal{N}=2$ E_7 superconformal theory. One can also examine the theory in the neighborhood of a zero of $f_{n+8}(z_2)$ of order k. Unlike the previous cases, here at a zero of order k one obtains k half-hypermultiplets (in the **56** representation). For k=1, and in general when one half-hypermultiplet is associated with a point in other cases, we will argue that the theory in the infrared is a new $\mathcal{N}=1$ fixed point.

At such a point, the curve (4.4) locally reduces to

$$y^2 = x^3 + z_1^3 z_2^k x + z_1^5 (4.6)$$

We assume that all terms in the curve (4.6) are equally relevant. Then, the by now familiar computation of dimensions gives $[z_1] = \frac{12k}{k+2}$ and $[z_2] = \frac{4}{k+2}$. For k = 1, we again obtain a new $\mathcal{N} = 1$ superconformal fixed point. The global symmetry group contains E_7 , and the gauge coupling on the Coulomb branch as one approaches the singularity is $\tau = e^{\frac{2\pi i}{3}}$.

For k > 1 and odd, the issues are much the same as the odd k, k > 1 case of §4.1. For k even it seems reasonable, from the description of Katz and Vafa [14], to conjecture that these theories flow to an $\mathcal{N} = 2$ E_8 fixed point. Note that no similar picture exists to describe matter in half-hypermultiplets, so it is reasonable to assume that the behavior of the brane probe would be different in these cases.

4.3. SO(K) at Spinor Points

One can analyze SO(K) theories at various special points by the same methods we have been using in previous sections. One sees from [10] that half spinor points occur on the SO(11) and SO(12) loci. By approaching such points, we expect to obtain new $\mathcal{N}=1$ superconformal theories whose global symmetry is at least SO(12) (and may be enhanced to E_7). For the SO(12) theory at a $\frac{1}{2}$ 32 point, we find dimensions $[z_1]=8/3$ and $[z_2]=4/3$. The coupling τ approaches $\tau=i$ near the singularity.

For the other SO(K) theories (or for the SO(11) and SO(12) cases when we merge an even number of half spinor points), we expect to get $\mathcal{N}=2$ theories as we approach points where spinors are localized. The computation of scaling dimensions gives results consistent with this expectation. For example, on the SO(10) locus by approaching 16 points one probably flows to the E_6 $\mathcal{N}=2$ superconformal theory. On the SO(8) locus at spinor points one flows to a (free) theory with SO(10) global symmetry (the SU(2) gauge theory with $N_f=5$).

We can also consider what happens when an SO(K) spinor point approaches a vector point. Let us do the analysis for SO(10) here. The curve at a vector+spinor point is

$$y^{2} = x^{3} + x(z_{1}^{2}z_{2}^{2} + z_{1}^{3} + \cdots) + (z_{1}^{3}z_{2}^{3} + z_{1}^{4}z_{2} + z_{1}^{5} + \cdots)$$

$$(4.7)$$

where here \cdots indicates terms of higher order in z_1, z_2 . Computing the dimensions, we find $[z_1] = 8/3$ and $[z_2] = 4/3$. Thus, this must also be a new nontrivial $\mathcal{N} = 1$ fixed point. Unlike the case with a single localized vector multiplet, there is no direct D-brane interpretation of the matter in this case along the lines of [14]. Other similar interesting possibilities exist but we will not discuss them here.

5. Discussion

By examining the behavior of threebrane probes in F-theory on elliptic Calabi-Yau threefolds, we have found evidence for the existence of many new nontrivial $\mathcal{N}=1$ supersymmetric renormalization group fixed points. In five cases we can actually prove, using the fact that z_2 is not a free field, that the theory in the infrared has only $\mathcal{N}=1$ supersymmetry. It is natural to think that in other cases (obtained by colliding various singularities) there are also new $\mathcal{N}=1$ fixed points. The five new fixed points we found which are definitely only $\mathcal{N}=1$ superconformal are:

- 1) The theory with SO(5) global symmetry group, obtained by approaching a monodromy point (on the z_2 plane) on the SO(5) locus in F-theory.
- 2) The theory with F_4 global symmetry, obtained by approaching a monodromy point on the F_4 locus in F-theory.
- 3) The theory with E_7 global symmetry which arises near a half **56** point on the E_7 locus in F-theory.
- 4) The theory obtained by approaching a half spinor point on the SO(12) locus.
- 5) The theory obtained by having a brane probe approach a vector + spinor point on the SO(10) locus.

It is possible that in many cases we discussed here where the dimensions are consistent with $\mathcal{N}=2$ supersymmetry, the theory nevertheless has only $\mathcal{N}=1$ supersymmetry in the IR (or flows to a new $\mathcal{N}=2$ fixed point). It is natural to conjecture that a systematic classification of all $\mathcal{N}=1$ superconformal theories with Coulomb phases by algebraic methods (such as the ones we used) may be possible.

It is interesting to note that the cases for which we have been able to rule out the possibility of $\mathcal{N}=2$ supersymmetry are exactly the cases for which we cannot interpret the worsening of the singularity as arising from a z_2 -dependent VEV for a spacetime field, as described in [14]. Thus, it is natural to speculate that whenever such a description does exist, the theory on the probe will actually flow to an $\mathcal{N}=2$ superconformal theory (with the appropriate enhanced global symmetry). The dimension of z_2 when we have such a description always comes out to equal one, so it can naturally become part of an adjoint "mass" parameter of the theory with the larger global symmetry.

It would be interesting to understand the Higgs branches of the theories we have found. We expect the Higgs branch of the theories with global symmetry G to correspond to (or at least to include) the moduli space of G-instantons on the seven branes. In the cases with Lagrangian descriptions, the Higgs phase corresponds to the space of Q and q VEVs modulo the D-term and superpotential constraints. Note that the superpotential may contain terms which lift part or all of the Higgs branch, without effecting our Coulomb branch analysis. Higher dimensional operators which have this effect may even be present at tree level in the field theory, and it is difficult to determine their exact form since we only have $\mathcal{N}=1$ supersymmetry.

These or similar $\mathcal{N}=1$ superconformal theories may be relevant for understanding what happens at singularities of 4d $\mathcal{N}=1$ supersymmetric string or F-theory compactifications. For example, SO(32) strings on Calabi-Yau threefolds manifest a nonperturbative

SU(2) gauge group at certain singularities in the moduli space; nonperturbative dynamics in the SU(2) explains the physics of the singularities [20]. It is natural to speculate that some classes of singularities of $E_8 \times E_8$ strings on Calabi-Yau threefolds will be explained by the appearance of a nontrivial fixed point theory.⁶

It would also be interesting to study brane probes obtained by wrapping a fivebrane on a higher genus curve in K3 compactifications of the heterotic string. At least for genus 2 the worldbrane quantum field theory should have interesting dynamics. Similarly, one could examine the behavior of N coincident threebrane probes [21], which should manifest an Sp(N) gauge theory on the worldvolume. At special points in the Coulomb branch of this theory, there may be new superconformal fixed points. More complicated singularities can also occur in F-theory compactifications [22]. The study of threebrane probes at these singularities should be interesting in its own right [21], and may also improve our understanding of the behavior of the spacetime theory in these cases.

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 $^{^{6}}$ We thank T. Banks for suggesting this possibility to us.

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